**Paper 01**

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**Question 01:**

**Proof:**

As we know that,

Then adding the corresponding sides of the inequalities:

(1)

Now we know that for a, b ϵ R:

Hence the equation 1 can be written as:

**Question 02:**

**Proof:**

We will break this proof into two parts.

1. Prove that there are infinitely many Fibonacci numbers.
2. Prove that the Fibonacci numbers grow exponentially.
3. Suppose there are finite Fibonacci numbers. Then there must exist the last number. We also know that the nth number in the Fibonacci numbers is always greater than all its previous numbers. Assuming that nth is the largest and last number of Fibonacci numbers. As we know that, for any nth Fibonacci number, the Fibonacci numbers (n-1)th and (n-2)th always exist as they occur before the nth. Now this nth and the (n-1)th can be added together to get the next Fibonacci number which is (n+1)th. This way we can always find next number of any nth number. But we said earlier that nth is the last number, which is contradicted by the (n+1)th. Thus, the Fibonacci numbers are infinitely many, because for every nth, we can get the (n+1)th.
4. If we take the ratio of adjacent Fibonacci numbers n­th and (n-1)th, as n grows larger, we get closer to the golden ratio i.e. 1.618. Thus, by the definition of a Geometric Sequence, the ratio r is greater than 1, so for any number, the next number is 1.618 times it. This proves that the Fibonacci numbers grow exponentially as we go farther into the sequence.

**Question 03:**

**Proof:**

We will prove this using the Prove by Mathematical Induction.

1. Base Case for n = 1

Putting n = 1 in the equation, we get,

[cos(ϴ) + i ∙ sin(ϴ)]1 = cos(1 ∙ ϴ) + i ∙ sin(1 ∙ ϴ)

cos(ϴ) + i ∙ sin(ϴ) = cos(ϴ) + i ∙ sin(ϴ)

which is true.

1. n = k

Supposing for n = k, the equation is true.

[cos(ϴ) + i ∙ sin(ϴ)]k = cos(kϴ) + i ∙ sin(kϴ)

1. n = k + 1

Now proving for n = k + 1.

Putting in the equation n = k + 1,

[cos(ϴ) + i ∙ sin(ϴ)]k + 1 = cos(kϴ + ϴ) + i ∙ sin(kϴ + ϴ)

Taking the R.H.S of the equation, using the cosine and sine rules for addition of angles.

cos(kϴ) cos(ϴ) – sin(kϴ) sin(ϴ) + i ∙ sin(kϴ) cos(ϴ) + i ∙ cos(kϴ) sin(ϴ)

cos(ϴ) [ cos(k ϴ) + i ∙ sin(k ϴ)] + i ∙ sin(ϴ) [cos(k ϴ) + i ∙ sin(k ϴ)]

[cos(ϴ) + i ∙ sin(ϴ)] [cos(k ϴ) + i ∙ sin(k ϴ)]

Now using the n = k equation,

[cos(ϴ) + i ∙ sin(ϴ)] [cos(ϴ) + i ∙ sin(ϴ)]k

[cos(ϴ) + i ∙ sin(ϴ)]k+1

This is equal to the L.H.S.

Thus, it holds true for n = k + 1.

Therefore, the given statement is true using Prove by Mathematical Induction.

**Question 04:**

**Proof:**

Suppose we have another prime triplet A = {p, p+2, p+4} that has a difference of 2.

From background knowledge we know that, all prime numbers are odd except 2. Now, we take the number 3, and divide each of the numbers in the set A.

1. p%3 = 0

If p has a factor 3, then it is not a prime number. Hence the original supposition is contradicted.

1. p%3 = 1

If p leaves a remainder of 1, then (p+2) will definitely leave a remainder of 0. Thus (p+2) has a factor 3 and is not a prime number. Hence the original supposition is contradicted.

1. p%3 = 2

If p leaves a remainder of 2, then (p+2) leave a remainder of 1 and (p+4) will definitely leave a remainder of 0. Thus (p+4) has a factor 3 and is not a prime number. Hence the original supposition is contradicted.

Hence, we cannot find a triplet that is prime and has a difference of 2.

**Question 05:**

**Proof:**

For our convenience, assume that tree is a directed graph with edges from child to parents. From background knowledge we conclude that:

1. Every vertex has exactly one parent except root node.
2. A tree has exactly one root node.
3. All the vertices are connected to their parents by exactly one edge.
4. No vertex is connected to any other vertex except its parent.

For a tree of n vertices, from (1) and (2), we conclude that there is n – 1 vertex with parents. From (3) we conclude that there is at least n – 1 edge. From (4) we conclude that there is no edge except those n – 1 edge. Hence, we conclude that, for a tree with n vertices, we have exactly n – 1 edge.

**Question 06:**

**Proof:**

Suppose an odd number p, such that it is prime.

We want to prove that the number p can be written as

p = a2 + b2, where a and b are any natural numbers

if and only if p ≡ 1 mod 4

We know that p is odd, hence, one of a2 or b2 is must odd and the other one is must even.

Now, only even can be a square of even number, and only odd can be a square of an odd number.

Supposing a2 is even and b2 is odd. Then,

a2 = (2m)2

b2 = (2n + 1)2

Now,

p = (2m)2 + (2n + 1)2

p = 4m2 + 4n2 + 4n +1

p = 4(m2 + n2 + n) + 1 ----- (1)

Thus, the number ‘p’ must leave a remainder of 1 when divided by 4 by equation (1). Thus,

p ≡ 1 mod 4 is true.

**Paper 02**

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**Question 01:**

|  |  |  |
| --- | --- | --- |
| **Rubric** | **Score** | **Reason** |
| Logical Correctness | 0 | The original statement is wrong, so it cannot be proven true. |
| Clarity | 2 | The proof is neat and clean, but some parts could be clearer. |
| Opening | 0 | There is no opening statement. |
| Stating the Conclusion | 4 | The conclusion is clear and concise, provided at the end of the proof. |
| Reasons | 0 | The reasoning is wrong. |
| Overall Valuation | 0 | The value of the proof is null as the person is correctly proving a wrong theorem. |

**Question 02:**

Part (b) **∀x∃y[Prime(x) ∧ Prime(y)∧(x<y)]** is correct.

**Reason:**

The reason is that for all-natural numbers x that are prime which is defined in the statement, there exist a natural number y that is prime which is also defined in the statement, the y is greater than number x.

**Question 03:**

Line Number 13.

If S(n) is true, then we need to prove S(n+1) using three people i.e. a, b and c. But for n = 1, even if S(1) is true, we cannot prove S(2) true using three people, as it contains two people.

**Question 04:**

**Part A:**

A clause *C* ≡ (*L*1 ∨...∨*Lk*), where the *Li*’s are literals *C* is valid if and only if, for some *i*, *j*∈{1,...,*k*}, *Li* ≡ ¬*Lj*.

**Case 01:**

If C is valid, then there exists i, j such that Li​≡¬Lj

This is not true because, If C is valid, there can be a case when all the literals are True. Hence, this is not true that If C is valid, then there exists i, j such that Li​≡¬Lj.

**Case 02:**

If for some i, j such that Li​≡¬Lj, C is valid.

This is true because if Li is True then Lj is False and vice versa. Hence, The C is valid.

Hence the original statement, which is a biconditional is not true that states, A clause C ≡ (L1 ∨...∨Lk), where the Li’s are literals C is valid if and only if, for some i, j∈{1,...,k}, Li ≡ ¬Lj

**Part B:**

The algorithm is:

**For each clause C:**

**For each literal L in C:**

**If there exists a complementary literal ¬L in the same clause C:**

**Return "Valid."**

**If no clause contains complementary literals:**

**Return "Invalid."**

This algorithm will check for each literal and return validity based on the condition that the negation of current literal exists in the clause.

**Part C:**

((A∨B) ⇒ C) ⇒ (A⇒C)

Using Implication formula, P ⇒ Q is equal to ¬ P ∨ Q,

¬((A∨B) ⇒ C) ∨ (¬A∨C)

Using De Morgan's Law and distribution:

¬(¬(A∨B) ∨ C) ∨ (¬A∨C)

Using De Morgan's Law

(A∨B∧¬C) ∨ (¬A∨C)

Simplifying

(A∨B) ∧ ((¬C∨¬A) ∨ C)

(A∨B) ∧ (¬A∨¬C∨C)

(A∨B) ∧ (¬A ∨ C)

**Validity:**

As the clause (A∨B) ∧ (¬A ∨ C) contains ¬A and A, hence it is valid.

**Question 05:**

**Part A:**

Let’s start by taking the second statement,

A ∨ ¬B ∨¬C ∨ D

Now rearranging,

(¬B ∨¬C) ∨ (A ∨ D)

Taking negation common from the first term

¬ (B C) ∨ (A ∨ D)

Writing it in the form of implication.

(B C) (A ∨ D)

**Part B:**

¬(X1 ∨···∨Xn) ≡ (¬X1 ∧···∧¬Xn).

1. General Case for n = 2

¬(X1 ∨ X2) = (¬X1 ∧ ¬X2)

Which holds true for any X1 and X2.

1. For n = k

Suppose the statement is true for n = k.

¬(X1 ∨···∨Xk) ≡ (¬X1 ∧···∧¬Xk)

Let A = (X1 ∨···∨Xk)

Then ¬A = ¬ (X1 ∨···∨Xk)

1. For n = k + 1

Now for k + 1 terms,

¬(X1 ∨···∨Xk ∨ Xk+1) ≡ (¬X1 ∧···∧¬Xk ∧¬Xk+1)

Taking the L.H.S of the equation,

¬(X1 ∨···∨Xk ∨ Xk+1)

Replacing A,

¬(A∨ Xk+1)

¬A ∧ ¬Xk+1

Substituting values,

¬ (X1 ∨···∨Xk) ∧ ¬Xk+1

(¬X1 ∧···∧¬Xk) ∧ ¬Xk+1

(¬X1 ∧···∧¬Xk ∧ ¬Xk+1)

Thus, it holds true for n = k + 1.

Therefore, the given statement is true using Prove by Mathematical Induction.

**Part C:**

In a Boolean expression where things are joined by "or," we have a special way of writing it (CNF). Each part of this expression is a bunch of things joined by "or." By applying a rule to these parts (De Morgan's Law), we get a different way of writing the expression (DNF). Applying the rule again brings us to another way of writing it (KNF). So, no matter how you start writing a Boolean expression, you can change it to this specific form (KNF), and it'll still mean the same thing.

**Question 06:**

**Part A:**

1. Pepperoni Sausage
2. Sausage Quail
3. (Pepperoni Quail) (Ricotta Cheese)

**Part B:**

The Geoff is a truth teller.

**Part C:**

The desired ingredients always include Ricotta Cheese. In case of Pepperoni, only cheese is required, and in case of Sausage, Quail will also be required with Cheese.

**Question 07:**

**Part A:**

1. Popcorn Raisin
2. Cucumber Sandwich Soda
3. Soda Tea)

**Part B:**

1. Popcorn Raisin
2. Cucumber Sandwich Soda
3. Soda Tea

**Part C:**

So, he either drinks Soda, or drinks Tea with Cucumber Sandwich.